AS-84.3127 Paikannus- ja navigointimenetelmät

Ratkaisut 1.

1)

Taylorin sarjakehitelmä: $f(x) = f(x_0) + \left.\frac{\partial f(x)}{\partial x}\right|_{x=x_0}(x - x_0) + h.o.t.$

Malli $\dot{x} = f(x)$ linearisoidaan toimintapisteessä x_0 säilyttämällä vain sarjan ensimmäiset termit. Eli merkitään $\delta\dot{x} = \dot{x} - \dot{x}_0$ ja $\delta x = x - x_0$, jolloin linearisoinniksi saadaan

$$\delta\dot{x} = f(x) - f(x_0) = \left.\frac{\partial f(x)}{\partial x}\right|_{x=x_0}\delta x \qquad (1)$$

Mikäli $f(x)$ ja x ovat vektoreita niin jakobiaani on matriisi muotoa

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m(x)}{\partial x_1} & \cdots & \frac{\partial f_m(x)}{\partial x_n} \end{bmatrix}. \qquad (2)$$

Ensimmäisessä tehtävässä piti laskea yhtälöryhmän

$$\begin{bmatrix} n \\ \dot{e} \\ \dot{v}_n \\ \dot{v}_e \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} v_n \\ v_e \\ \cos(\psi)\tilde{a}_u - \sin(\psi)\tilde{a}_v \\ \sin(\psi)\tilde{a}_u + \cos(\psi)\tilde{a}_u \\ \tilde{\omega}_r \end{bmatrix} = \begin{bmatrix} v_n \\ v_e \\ \cos(\psi)a_u - \sin(\psi)a_v \\ \sin(\psi)a_u + \cos(\psi)a_u \\ \omega_r \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta a_u \\ \delta a_v \\ \delta\omega_r \end{bmatrix}$$

linearisointi. Yhtälöryhmän linearisoimiseksi tarvitaan osittaisderivaattamatriisi eli

$$
\begin{bmatrix}
\frac{\partial v_n}{\partial n} & \frac{\partial v_n}{\partial e} & \frac{\partial v_n}{\partial v_n} & \frac{\partial v_n}{\partial v_e} & \frac{\partial v_n}{\partial \psi} \\
\frac{\partial v_e}{\partial n} & \frac{\partial v_e}{\partial e} & \frac{\partial v_e}{\partial v_n} & \frac{\partial v_e}{\partial v_e} & \frac{\partial v_e}{\partial \psi} \\
\frac{\partial a_n}{\partial n} & \frac{\partial a_n}{\partial e} & \frac{\partial a_n}{\partial v_n} & \frac{\partial a_n}{\partial v_e} & \frac{\partial a_n}{\partial \psi} \\
\frac{\partial a_e}{\partial n} & \frac{\partial a_e}{\partial e} & \frac{\partial a_e}{\partial v_n} & \frac{\partial a_e}{\partial v_e} & \frac{\partial a_e}{\partial \psi} \\
\frac{\partial \omega_r}{\partial n} & \frac{\partial \omega_r}{\partial e} & \frac{\partial \omega_r}{\partial v_n} & \frac{\partial \omega_r}{\partial v_e} & \frac{\partial \omega_r}{\partial \psi}
\end{bmatrix}
=
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -\sin(\psi) a_u - \cos(\psi) a_v \\
0 & 0 & 0 & 0 & \cos(\psi) a_u - \sin(\psi) a_v \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
=
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -a_e \\
0 & 0 & 0 & 0 & a_n \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\tag{4}
$$

Kun tämä sijoitetaan kaavaan (1) saadaan

$$
\begin{bmatrix}
\delta \dot{n} \\ \delta \dot{e} \\ \delta \dot{v}_n \\ \delta \dot{v}_e \\ \delta \dot{\psi}
\end{bmatrix}
=
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -a_e \\
0 & 0 & 0 & 0 & a_n \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\delta n \\ \delta e \\ \delta v_n \\ \delta v_e \\ \delta \psi
\end{bmatrix}
+
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\cos(\psi) & -\sin(\psi) & 0 \\
\sin(\psi) & \cos(\psi) & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\delta a_u \\ \delta a_v \\ \delta \omega_r
\end{bmatrix}
$$

mikä oli myös vastaus.

2)

Malli on muotoa

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w \qquad (5)$$

Tässä olemme kiinnostuneita siis pelkästään virhetermin vaikutuksesta.

a)

jos virhetermi on vakio, saadaan matkalle yhtälö $\ddot{x} = w$, $x(0) = 0$. Kyseiselle yhtälölle saa intergoimalla puolittain ratkaisun $x = \frac{1}{2} wt^2$.

b)

Lineaarisen tilayhtälön tilan kovarianssin muuttumista kuvaa Riccatin yhtälö

$$\dot{P} = AP + PA^T + Q$$

missä P on tilan kovarianssimatriisi, A on tilansiirtomatriisi joka tässä tapauksessa saadaan kaavasta (5) ja Q on kohinan kovarianssimatriisi.

Eli $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ ja $Q = E\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} ww \begin{bmatrix} 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ 0 & Q_{22} \end{bmatrix}$

missä Q_{22} on *w*:n varianssi.

Saadaan

$$\begin{bmatrix} \dot{p}_1 & \dot{p}_2 \\ \dot{p}_3 & \dot{p}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & Q_{22} \end{bmatrix}$$

ja edelleen

$$\begin{bmatrix} \dot{p}_1 & \dot{p}_2 \\ \dot{p}_3 & \dot{p}_4 \end{bmatrix} = \begin{bmatrix} p_3 + p_2 & p_4 \\ p_4 & Q_{22} \end{bmatrix}$$

itse asiassa kovarianssimatriisi on symmetrinen eli p_2 ja p_3 ovat yhtäsuuria, kun tämä otetaan huomioon saadaan kolme differentiaaliyhtälöä.

$$\dot{p}_1 = 2p_2$$
$$\dot{p}_2 = p_4$$

ja

$$\dot{p}_4 = Q_{22}$$

Ratkaistaan alkuarvolla $P = 0$:

$p_4 = Q_{22} * t$,

$$p_2 = \frac{1}{2} Q_{22} * t^2$$

ja

$$p_1 = \frac{1}{3} Q_{22} * t^3$$

eli virheen keskipoikkeamaksi ajanhetkellä t saadaan $\sqrt{p_1} = \frac{1}{\sqrt{3}} \sqrt{Q_{22}} * t^{1.5}$.

AS-84.3127 Localization and navigation methods

Solutions for exercise 1.

1)

Taylor series expansion: $f(x) = f(x_0) + \left.\frac{\partial f(x)}{\partial x}\right|_{x=x_0}(x - x_0) + h.o.t.$

The model $\dot{x} = f(x)$ is linearized in the operation point x_0 by taking the first terms from the Taylor expansion. Let's mark $\delta\dot{x} = \dot{x} - \dot{x}_0$ and $\delta x = x - x_0$ then the linearized model is given as

$$\delta\dot{x} = f(x) - f(x_0) = \left.\frac{\partial f(x)}{\partial x}\right|_{x=x_0}\delta x \qquad (1)$$

In case $f(x)$ and x are vectors then the Jacobian has the following matrix form

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m(x)}{\partial x_1} & \cdots & \frac{\partial f_m(x)}{\partial x_n} \end{bmatrix}. \qquad (2)$$

In the first problem the following group of equations had to be linearized:

$$\begin{bmatrix} n \\ \dot{e} \\ \dot{v}_n \\ \dot{v}_e \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} v_n \\ v_e \\ \cos(\psi)\tilde{a}_u - \sin(\psi)\tilde{a}_v \\ \sin(\psi)\tilde{a}_u + \cos(\psi)\tilde{a}_u \\ \tilde{\omega}_r \end{bmatrix} = \begin{bmatrix} v_n \\ v_e \\ \cos(\psi)a_u - \sin(\psi)a_v \\ \sin(\psi)a_u + \cos(\psi)a_u \\ \omega_r \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \delta a_u \\ \delta a_v \\ \delta\omega_r \end{bmatrix}$$

For the linearization the matrix of partial derivatives is required

$$\begin{bmatrix} \frac{\partial v_n}{\partial n} & \frac{\partial v_n}{\partial e} & \frac{\partial v_n}{\partial v_n} & \frac{\partial v_n}{\partial v_e} & \frac{\partial v_n}{\partial \psi} \\ \frac{\partial v_e}{\partial n} & \frac{\partial v_e}{\partial e} & \frac{\partial v_e}{\partial v_n} & \frac{\partial v_e}{\partial v_e} & \frac{\partial v_e}{\partial \psi} \\ \frac{\partial a_n}{\partial n} & \frac{\partial a_n}{\partial e} & \frac{\partial a_n}{\partial v_n} & \frac{\partial a_n}{\partial v_e} & \frac{\partial a_n}{\partial \psi} \\ \frac{\partial a_e}{\partial n} & \frac{\partial a_e}{\partial e} & \frac{\partial a_e}{\partial v_n} & \frac{\partial a_e}{\partial v_e} & \frac{\partial a_e}{\partial \psi} \\ \frac{\partial \omega_r}{\partial n} & \frac{\partial \omega_r}{\partial e} & \frac{\partial \omega_r}{\partial v_n} & \frac{\partial \omega_r}{\partial v_e} & \frac{\partial \omega_r}{\partial \psi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\sin(\psi) a_u - \cos(\psi) a_v \\ 0 & 0 & 0 & 0 & \cos(\psi) a_u - \sin(\psi) a_v \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -a_e \\ 0 & 0 & 0 & 0 & a_n \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4}$$

Assigning this to equation (1) yields

$$\begin{bmatrix} \delta \dot{n} \\ \delta \dot{e} \\ \delta \dot{v}_n \\ \delta \dot{v}_e \\ \delta \dot{\psi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -a_e \\ 0 & 0 & 0 & 0 & a_n \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta n \\ \delta e \\ \delta v_n \\ \delta v_e \\ \delta \psi \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta a_u \\ \delta a_v \\ \delta \omega_r \end{bmatrix}$$

which is also the answer.

2)

The model has the following form

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w \qquad (5)$$

Here we are only interested in effects of the error term.

a)

if the error term is constant, we get the following equation for the traveled distance. $\ddot{x} = w$, $x(0) = 0$. By integrating both sides the following solution is given for the equation $x = \frac{1}{2} wt^2$.

b)

The change of the covariance of a linear state equation is described with the Riccati equation

$$\dot{P} = AP + PA^T + Q$$

where P is the state covariance matrix, A is the state dynamics matrix, which in this case is derived from equation (5) and Q is the covariance matrix of the uncertainty involved in the state dynamics equation.

Consequently $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $Q = E\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} ww \begin{bmatrix} 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ 0 & Q_{22} \end{bmatrix}$

Where Q_{22} is the variance of *w*.

Now the following equation is acquired

$$\begin{bmatrix} \dot{p}_1 & \dot{p}_2 \\ \dot{p}_3 & \dot{p}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & Q_{22} \end{bmatrix}$$

and further

$$\begin{bmatrix} \dot{p}_1 & \dot{p}_2 \\ \dot{p}_3 & \dot{p}_4 \end{bmatrix} = \begin{bmatrix} p_3 + p_2 & p_4 \\ p_4 & Q_{22} \end{bmatrix}$$

actually, the covariance matrix is symmetric i.e. the terms p_2 and p_3 are identical, which gives the following three differential equations.

$$\dot{p}_1 = 2p_2$$
$$\dot{p}_2 = p_4$$

and

$$\dot{p}_4 = Q_{22}$$

Solve by assigning the initial value $P = 0$:

$$p_4 = Q_{22} * t,$$
$$p_2 = \frac{1}{2} Q_{22} * t^2$$

and

$$p_1 = \frac{1}{3} Q_{22} * t^3$$

which gives the standard deviation of the error at time t: $\sqrt{p_1} = \frac{1}{\sqrt{3}} \sqrt{Q_{22}} * t^{1.5}$.